Journal of Sound and Vibration (1999) **227**(5), 1122–1132
Article No. jsvi.1999.2375, available online at http://www.idealibrary.com on IDEAL®

COMPARING THE PERTURBED EIGENSOLUTIONS OF A GENERALIZED AND A STANDARD EIGENVALUE PROBLEM

PHILIP D. CHA* AND WEIQING GU†

Department of Engineering, Harvey Mudd College, Claremont, CA 91711, *U.S.A.*

(*Received* 10 *May* 1999)

1. INTRODUCTION

A frequently encountered scenario in structural dynamics is determining the changes in the eigensolution of a system after certain modifications are introduced. Clearly, if these modifications are substantial, then a new analysis and computational cycle are necessary in order to compute the new eigendata. However, if the changes made are small, then the perturbation theory can be applied whereby the initial modal characteristics are used as a basis to extract the new eigensolution of the modified system without performing a new and possibly costly analysis. Over the years the perturbation theory has been used in the solution of many different problems, and hence only a few selected references are given [1–7].

In this technical note, the perturbation theory will be used to determine the first order eigensolutions of a slightly perturbed symmetric generalized eigenvalue problem and its corresponding standard eigenvalue problem. The first order perturbation results obtained in this technical note are well known and certainly not new. The objective of this technical note is not to show how the perturbation theory can be applied to extract the modes of vibration of slightly modified structures. Instead, the goal is to highlight the similarities and differences in the first order perturbed eigensolutions for the same system obtained by solving a generalized eigenvalue problem and its corresponding standard eigenvalue problem. In particular, it will be shown that a certain coefficient that is commonly assumed zero (in the standard eigenvalue problem formulation) cannot be neglected. Numerical examples and comparisons will be made to illustrate the importance of this term.

*Associate Professor of Engineering, Harvey Mudd College, Claremont, CA, U.S.A.
†Assistant Professor of Mathematics, Harvey College, Claremont, CA, U.S.A.

0022-460X/99/451122 + 11 $30.00/0

1.1. SYMMETRIC GENERALIZED EIGENVALUE PROBLEM

Consider a system whose initial free response is governed by

$$[K_o]\mathbf{x}_{oj} = \lambda_{oj}[M_o]\mathbf{x}_{oj}, \tag{1}$$

where $[K_o]$ and $[M_o]$ are the real, symmetric unperturbed stiffness and mass matrices of size $N \times N$, whose jth eigenvalue and eigenvector are denoted by λ_{oj} and $\mathbf{x}_{oj}$, respectively. Assume the N eigenvalues are all distinct and the eigenvectors are properly normalized such that the following orthogonality conditions are met:

$$\mathbf{x}_{oj}^{\mathrm{T}}[M_o]\mathbf{x}_{oi} = \delta_i^j, \qquad \mathbf{x}_{oj}^{\mathrm{T}}[K_o]\mathbf{x}_{oi} = \lambda_{oj}\delta_i^j, \tag{2}$$

where δ_i^j is the Kronecker delta.

The system is now slightly modified or perturbed such that its new modes of vibration are obtained by solving

$$[K]\mathbf{x}_j = \lambda_j[M]\mathbf{x}_j, \tag{3}$$

where matrices $[K]$ and $[M]$ remain symmetric. The perturbed stiffness and mass matrices are given by

$$[K] = [K_o] + [\delta K] + \cdots, \qquad [M] = [M_o] + [\delta M] + \cdots, \tag{4}$$

where $[\delta K]$ and $[\delta M]$ correspond to the symmetric first order perturbation matrices. Assuming the modifications made are small, then the eigensolution of equation (1) can be used to derive approximate expressions for the eigensolution of equation (3). To this end, the eigenvalues and eigenvectors of equation (3) are assumed to differ from those of equation (1) by some small perturbations as follows:

$$\lambda_j = \lambda_{oj} + \delta\lambda_j + \cdots, \qquad \mathbf{x}_j = \mathbf{x}_{oj} + \delta\mathbf{x}_j + \cdots, \tag{5}$$

where $(\lambda_{oj}, \mathbf{x}_{oj})$ are the jth unperturbed eigensolution, and $(\delta\lambda_j, \delta\mathbf{x}_j)$ are the jth first order perturbations. The unperturbed eigenvectors are normalized according to equation (2). Thus, the unperturbed eigenvectors, the $\mathbf{x}_{oj}$'s, form a complete orthonormal set (with respect to $[M_o]$) in the N-dimensional space, and any vector in that N-dimensional space may be expressed as a linear combination of the $\mathbf{x}_{oj}$'s. Hence, the jth first order eigenvector perturbation can be written as

$$\delta\mathbf{x}_j = \sum_{r=1}^{N} \varepsilon_{rj}\mathbf{x}_{or}, \tag{6}$$

where the ε_{rj}'s are the small coefficients to be determined.

Substituting equations (5) and (4) into equation (3) and keeping only the first order terms, we obtain

$$[K_o]\delta\mathbf{x}_j + [\delta K]\mathbf{x}_{oj} = \lambda_{oj}[M_o]\delta\mathbf{x}_j + \lambda_{oj}[\delta M]\mathbf{x}_{oj} + \delta\lambda_j[M_o]\mathbf{x}_{oj}. \tag{7}$$

Substituting equation (6) into equation (7) yields

$$[K_o]\sum_{r=1}^{N} \varepsilon_{rj}\mathbf{x}_{or} + [\delta K]\mathbf{x}_{oj} = \lambda_{oj}[M_o]\sum_{r=1}^{N} \varepsilon_{rj}\mathbf{x}_{or} + \lambda_{oj}[\delta M]\mathbf{x}_{oj} + \delta\lambda_j[M_o]\mathbf{x}_{oj}. \tag{8}$$

Premultiplying equation (8) by $\mathbf{x}_{oj}^{\mathrm{T}}$ and recalling the orthogonality conditions of equation (2), we have

$$\delta\lambda_j = \mathbf{x}_{oj}^{\mathrm{T}}([\delta K] - \lambda_{oj}[\delta M])\mathbf{x}_{oj}, \tag{9}$$

Premultiplying equation (8) by $\mathbf{x}_{oj}^{\mathrm{T}}(i \neq j)$ and recalling the orthogonality conditions of equation (2), we get

$$\varepsilon_{ij} = \frac{\mathbf{x}_{oi}^{\mathrm{T}}([\delta K] - \lambda_{oj}[\delta M])\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}. \tag{10}$$

To determine the coefficient ε_{jj}, we assume the perturbed eigenvectors, the $\mathbf{x}_j$'s, satisfy the orthogonality condition

$$\mathbf{x}_j^{\mathrm{T}}[M]\mathbf{x}_j = 1. \tag{11}$$

After some algebra, we obtain

$$\varepsilon_{jj} = -\tfrac{1}{2}\mathbf{x}_{oj}^{\mathrm{T}}[\delta M]\mathbf{x}_{oj}. \tag{12}$$

Thus, to the first order, the jth perturbed eigenvalue and eigenvector are given by

$$\lambda_j = \lambda_{oj} + \mathbf{x}_{oj}^{\mathrm{T}}([\delta K] - \lambda_{oj}[\delta M])\mathbf{x}_{oj}, \tag{13}$$

$$\mathbf{x}_j = \mathbf{x}_{oj}(1 - \tfrac{1}{2}\mathbf{x}_{oj}^{\mathrm{T}}[\delta M]\mathbf{x}_{oj}) + \sum_{i=1, i \neq j}^{N} \frac{\mathbf{x}_{oi}^{\mathrm{T}}([\delta K] - \lambda_{oj}[\delta M])\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}\mathbf{x}_{oi}. \tag{14}$$

1.2. NON-SYMMETRIC STANDARD EIGENVALUE PROBLEM

It is often desirable to solve a standard eigenvalue problem as opposed to a generalized eigenvalue problem. The rationale is two-fold. First, many numerically efficient algorithms have been developed over the years specifically for the solution of a standard eigenvalue problem [8–10]. Second, the behavior of a standard eigenvalue problem is much better understood, and many theorems have been formulated for a standard as opposed to a generalized eigenvalue problem [11]. Motivated by the above arguments, the perturbation theory is also applied to the standard eigenvalue problem that corresponds to equation (3).

If $[M_o]$ is positive definite, it is always possible to convert equation (1) into the standard eigenvalue problem.

$$[A_o]\mathbf{x}_{oj} = \lambda_{oj}\mathbf{x}_{oj}, \tag{15}$$

where the λ_{oj}'s are assumed to be distinct and

$$[A_o] = [M_o]^{-1}[K_o]. \tag{16}$$

It is important to note that even $[M_o] = [M_o]^{\mathrm{T}}$ and $[K_o] = [K_o]^{\mathrm{T}}$, $[A_o]$ is generally *not* symmetric. The symmetric generalized eigenvalue problem of equation (1) can also be written as

$$[K_o][M_o]^{-1}[M_o]\mathbf{x}_{oj} = \lambda_{oj}[M_o]\mathbf{x}_{oj}. \tag{17}$$

Noting equation (16) and introducing the co-ordinate transformation

$$\mathbf{y}_{oj} = [M_o]\mathbf{x}_{oj}, \tag{18}$$

equation (17) reduces to

$$[A_o]^{\mathrm{T}}\mathbf{y}_{oj} = \lambda_{oj}\mathbf{y}_{oj}, \tag{19}$$

which is known as the adjoint eigenvalue problem associated with equation (15). Since $[M_o]$ is invertible and since the set of eigenvectors $(\mathbf{x}_{oj})_{j=1,\dots,N}$ forms a basis in the N-dimensional space, then the set of $(\mathbf{y}_{oj})_{j=1,\dots,N}$ also forms a basis in the same N-dimensional space. The eigenvectors $\mathbf{x}_{oj}$ and $\mathbf{y}_{oj}$ are also known as the right and left eigenvectors of $[A_o]$, respectively. For convenience, $\mathbf{x}_{oj}$ and $\mathbf{y}_{oj}$ are normalized so that they satisfy

$$\mathbf{y}_{oj}^{\mathrm{T}}\mathbf{x}_{oi} = \delta_i^j, \qquad \mathbf{y}_{oj}^{\mathrm{T}}[A_o]\mathbf{x}_{oi} = \lambda_{oj}\delta_i^j. \tag{20}$$

Substituting equation (18) into equation (20), we recover the orthogonality conditions of equation (2).

When the system is slightly perturbed, equation (3) can always be manipulated (provided $[M]$ is also positive definite) into the standard eigenvalue problem.

$$[A]\mathbf{x}_j = \lambda_j\mathbf{x}_j, \tag{21}$$

where $[A]$ is generally non-symmetric of the form

$$[A] = [M]^{-1}[K]. \tag{22}$$

Matrix $[A]$ can be expressed as

$$[A] = [A_o] + [\delta A] + \cdots, \tag{23}$$

where $[A_o]$ is given by equation (16) and $[\delta A]$ represents the first order perturbation matrix. Equation (22) can also be written as

$$[M][A] = [K] \tag{24}$$

or

$$([M_o] + [\delta M] + \cdots)([A_o] + [\delta A] + \cdots) = [K_o] + [\delta K] + \cdots. \tag{25}$$

Expanding equation (25) and retaining only the first order terms, we find the following expression for $[\delta A]$:

$$[\delta A] = [M_o]^{-1}([\delta K] - [\delta M][M_o]^{-1}[K_o]). \tag{26}$$

Substituting equations (5) and (23) into equation (21) and keeping only the first order terms, we obtain

$$[A_o]\delta\mathbf{x}_j + [\delta A]\mathbf{x}_{oj} = \lambda_{oj}\delta\mathbf{x}_j + \delta\lambda_j\mathbf{x}_{oj}. \tag{27}$$

Substituting equation (6) into equation (27), premultiplying the resultant expression by $\mathbf{y}_{oj}^{\mathrm{T}}$ and recalling the orthogonality conditions of equation (20), we get

$$\delta\lambda_j = \mathbf{y}_{oj}^{\mathrm{T}}[\delta A]\mathbf{x}_{oj}. \tag{28}$$

Utilizing the same procedure outlined above except premultiplying by $\mathbf{y}_{oi}^{\mathrm{T}}$ $(i \neq j)$, we find

$$\varepsilon_{ij} = \frac{\mathbf{y}_{oi}^{\mathrm{T}}[\delta A]\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}. \tag{29}$$

In the expansion of equation (6) for the non-symmetric standard eigenvalue problem of equation (21), Meirovitch [12] assumed that $\varepsilon_{jj} = 0$ from the onset. He claimed that this is done to guarantee $\delta\mathbf{x}_i = \mathbf{0}$ when $[\delta A] = [0]$, and to ensure the coefficient of $\mathbf{x}_{oi}$ remains equal to 1 when $[A_o]$ is replaced by $[A]$. However, since ε_{ij} for $i \neq j$ is retained in the first order approximation, it is unclear why ε_{jj} has to vanish.

Pierre [13] also carried out a similar perturbation analysis. He determined that $\varepsilon_{jj} = 0$ by keeping the inner product of the left and right eigenvectors a constant as follows:

$$\mathbf{y}_j^{\mathrm{T}}\mathbf{x}_j = (\mathbf{y}_{oj} + \delta\mathbf{y}_j)^{\mathrm{T}}(\mathbf{x}_{oj} + \delta\mathbf{x}_j) = 1. \tag{30}$$

Expanding equation (30) and keeping only the first order terms leads to

$$\mathbf{y}_{oj}^{\mathrm{T}}\delta\mathbf{x}_j + \mathbf{x}_{oj}^{\mathrm{T}}\delta\mathbf{y}_j = 0. \tag{31}$$

Assuming that the eigenvectors can be perturbed symmetrically, i.e., assuming that

$$\mathbf{x}_{oj}^{\mathrm{T}}\delta\mathbf{y}_j = \mathbf{y}_{oj}^{\mathrm{T}}\delta\mathbf{x}_j, \tag{32}$$

he reduced equation (31) to

$$2\mathbf{y}_{oj}^{\mathrm{T}}\delta\mathbf{x}_j = 0. \tag{33}$$

Using equation (6) and the orthogonality condition of equation (20), he readily obtained $\varepsilon_{jj} = 0$. The above derivation, however, clearly hinges on the "symmetric perturbation" assumption, which appears arbitrary.

Return now to equation (31). The jth left eigenvector perturbation can be expressed as a linear combination of the unperturbed left eigenvectors as

$$\delta\mathbf{y}_j = \sum_{r=1}^{N} \alpha_{rj}\mathbf{y}_{or}, \tag{34}$$

where the α_{rj}'s are small constant coefficients. Substituting both equation (6) and (34) into equation (31) yields

$$\mathbf{y}_{oj}^{\mathrm{T}} \sum_{r=1}^{N} \varepsilon_{rj}\mathbf{x}_{or} + \mathbf{x}_{oj}^{\mathrm{T}} \sum_{r=1}^{N} \alpha_{rj}\mathbf{y}_{or} = \varepsilon_{jj} + \alpha_{jj} = 0. \tag{35}$$

While ε_{jj} and α_{jj} sum to zero, they certainly do not have to vanish simultaneously. Only when $[A_o]$ is symmetric does the above lead to $\varepsilon_{jj} = 0$ (since $\alpha_{jj} = \varepsilon_{jj}$). For this particular case, the unperturbed system becomes self-adjoint, and without loss of generality, we can assume that the right and left unperturbed eigensectors coincide. Hence $\varepsilon_{jj} = 0$ is strictly valid for the above special case, and in general, $\varepsilon_{jj} \neq 0$.

Finally, using a first order perturbation analysis and assuming that $\varepsilon_{jj} = 0$, the jth perturbed eigenvalue and eigenvector for the non-symmetric standard eigenvalue problem of equation (21) can be expressed as

$$\lambda_j = \lambda_{oj} + \mathbf{y}_{oj}^{\mathrm{T}}[\delta A]\mathbf{x}_{oj}, \qquad \mathbf{x}_j = \mathbf{x}_{oj} + \sum_{i=1,\, i \neq j}^{N} \frac{\mathbf{y}_{oi}^{\mathrm{T}}[\delta A]\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}\, \mathbf{x}_{oi}. \tag{36, 37}$$

In contrast to equation (14), note that $\mathbf{x}_{oj}$ in equation (37) is not perturbed at all. Finally, the jth left eigenvector, $\mathbf{y}_j$, can be obtained in a similar fashion so that

$$\mathbf{y}_j = \mathbf{y}_{oj} + \sum_{i=1,\, i \neq j}^{N} \frac{\mathbf{x}_{oi}^{\mathrm{T}}[\delta A]^{\mathrm{T}}\mathbf{y}_{oj}}{\lambda_{oj} - \lambda_{oi}}\, \mathbf{y}_{oi}. \tag{38}$$

2. RESULTS

The perturbation expressions of equations (13)–(14) and equations (36)–(37) are obtained from the same symmetric generalized eigenvalue problem of equation (3). While they may appear different, they must somehow be related. Our goal now is to compare these results analytically and numerically.

2.1. ANALYTICAL COMPARISON

We first consider the perturbed eigenvalue expressions of equations (13) and (36). Substituting equation (18) and (26) into equation (36), we get

$$\lambda_j = \lambda_{oj} + ([M_o]\mathbf{x}_{oj})^{\mathrm{T}}[M_o]^{-1}([[\delta K] - [\delta M][M_o]^{-1}[K_o]])\mathbf{x}_{oj}. \tag{39}$$

Recalling equation (16), we have

$$\lambda_j = \lambda_{oj} + \mathbf{x}_{oj}^{\mathrm{T}}([\delta K]\mathbf{x}_{oj} - [\delta M][A_o]\mathbf{x}_{oj}). \tag{40}$$

Substituting equation (15) into the equation above, we obtain

$$\lambda_j = \lambda_{oj} + \mathbf{x}_{oj}^{\mathrm{T}}([\delta K] - \lambda_{oj}[\delta M])\mathbf{x}_{oj}, \tag{41}$$

which clearly shows that equation (13) and (36) are in fact identical.

We now proceed to compare the perturbed eigenvector expressions of equations (14) and (37). Substituting equations (18) and (26) into equation (37), we get

$$\mathbf{x}_j = \mathbf{x}_{oj} + \sum_{i=1,\, i \neq j}^{N} \frac{([M_o]\mathbf{x}_{oi})^{\mathrm{T}}[M_o]^{-1}([\delta K] - [\delta M][M_o]^{-1}[K_o])\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}\, \mathbf{x}_{oi}, \tag{42}$$

which can also be expressed as

$$\mathbf{x}_j = \mathbf{x}_{oj} + \sum_{i=1,\, i \neq j}^{N} \frac{\mathbf{x}_{oi}^{\mathrm{T}}([\delta K] - [\delta M][A_o])\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}\, \mathbf{x}_{oi} \tag{43}$$

or

$$\mathbf{x}_j = \mathbf{x}_{oj} + \sum_{i=1,\, i \neq j}^{N} \frac{\mathbf{x}_{oi}^{\mathrm{T}}([\delta K] - \lambda_{oj}[\delta M])\mathbf{x}_{oj}}{\lambda_{oj} - \lambda_{oi}}\, \mathbf{x}_{oi}. \tag{44}$$

Comparing equation (14) and (44), we immediately notice the absence of the term

$$(-\tfrac{1}{2}\mathbf{x}_{oj}^{\mathrm{T}}[\delta M]\mathbf{x}_{oj})\mathbf{x}_{oj} \tag{45}$$

in the perturbed eigenvector expression for the standard eigenvalue problem, which clearly corresponds to the first order perturbation term of $\varepsilon_{jj}\mathbf{x}_{oj}$. Thus, while both perturbation approaches lead to the same perturbed eigenvalues, they differ in the perturbed eigenvectors by a first order term given by equation (45).

We will now prove, by contradiction, whether the assumption of $\varepsilon_{jj}=0$ can be made. From equation (13), we noticed that

$$\delta\lambda_j = \mathbf{x}_{oj}^{\mathrm{T}}([\delta K]-\lambda_{oj}[\delta M])\mathbf{x}_{oj} = \mathbf{x}_{oj}^{\mathrm{T}}[\delta K]\mathbf{x}_{oj} - \lambda_{oj}\mathbf{x}_{oj}^{\mathrm{T}}[\delta M]\mathbf{x}_{oj}. \tag{46}$$

For $\varepsilon_{jj}=0$, equation (12) dictates that

$$\mathbf{x}_{oj}^{\mathrm{T}}[\delta M]\mathbf{x}_{oj}=0 \tag{47}$$

and equation (46) reduces to

$$\delta\lambda_j = \mathbf{x}_{oj}^{\mathrm{T}}[\delta K]\mathbf{x}_{oj} = \mathbf{x}_{oj}^{\mathrm{T}}([K]-[K_o])\mathbf{x}_{oj} = \mathbf{x}_{oj}^{\mathrm{T}}[K]\mathbf{x}_{oj} - \mathbf{x}_{oj}^{\mathrm{T}}[K_o]\mathbf{x}_{oj}. \tag{48}$$

Recalling equation (1), we can rewrite equation (48) alternatively as

$$\delta\lambda_j = \mathbf{x}_{oj}^{\mathrm{T}}[K]\mathbf{x}_{oj} - \lambda_{oj}\mathbf{x}_{oj}^{\mathrm{T}}[M_o]\mathbf{x}_{oj} = \mathbf{x}_{oj}^{\mathrm{T}}[K]\mathbf{x}_{oj} - \lambda_{oj}, \tag{49}$$

which yields

$$\lambda_j = \lambda_{oj} + \delta\lambda_j = \mathbf{x}_{oj}^{\mathrm{T}}[K]\mathbf{x}_{oj}. \tag{50}$$

Equation (50) implies that by assuming $\varepsilon_{jj}=0$, λ_j depends only on $[K]$ and not on $[M]$, contrary to the generalized eigenvalue problem formulation of equation (3), which clearly depends on both $[K]$ and $[M]$. Since the standard eigenvalue problem of equation (21) is derived from the generalized eigenvalue problem of equation (3), we have shown indirectly that the assumption of $\varepsilon_{jj}=0$ is generally not valid.

2.2. NUMERICAL COMPARISON

Having analytically compared the perturbed eigensolution of a symmetric generalized eigenvalue problem and its corresponding standard eigenvalue problem, we now numerically compare the perturbation results with the exact solution. The system under consideration is shown in Figure 1, where the mass matrix is diagonal and the stiffness matrix is tri-diagonal. The nominal system parameters and their perturbations are given in Table 1.

Equations (13) and (36) are used to compute the perturbed eigenvalues. Since the eigenvalues are scalar quantities, a simple percentage error relative to the exact eigenvalue can be used to compare the accuracy of the perturbed eigenvalues. Table 2 shows the exact and the first order perturbed eigenvalues. Also listed are the corresponding percentage errors relative to the exact eigenvalues. Note that the results of equations (13) and (36) are identical, and they are in excellent agreement with the exact solution.

We now turn our attention to the perturbed eigenvectors. Comparing the first order eigenvector expressions of equations (14)–(37), we immediately notice the absence of the $\varepsilon_{jj}\mathbf{x}_{oj}$ term in equation (37). Of great interest is the effect of this term

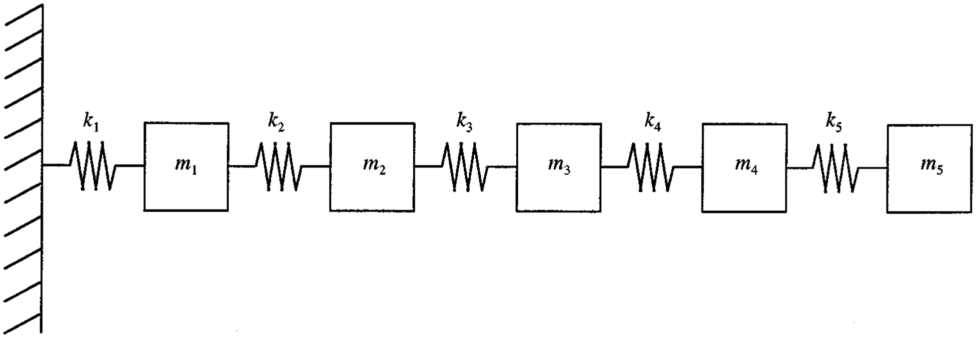

Figure 1. Simple chain of coupled oscillators.

TABLE 1

The nomial system parameters and their perturbations for the system of Figure 1. The last column represents the percentage deviation from the nominal value

Nominal	Perturbations	% Deviation
$m_1 = 5045$ kg	$\delta m_1 = -443$ kg	− 8·78
$m_2 = 5045$ kg	$\delta m_2 = 648$ kg	12·84
$m_3 = 5045$ kg	$\delta m_3 = 148$ kg	2·93
$m_4 = 5045$ kg	$\delta m_4 = -738$ kg	− 14·63
$m_5 = 3810$ kg	$\delta m_5 = 341$ kg	8·95
$k_1 = 1{\cdot}622 \times 10^6$ N/m	$\delta k_1 = -1{\cdot}63 \times 10^5$ N/m	− 10·05
$k_2 = 1{\cdot}622 \times 10^6$ N/m	$\delta k_2 = 9{\cdot}60 \times 10^4$ N/m	5·92
$k_3 = 1{\cdot}622 \times 10^6$ N/m	$\delta k_3 = 2{\cdot}10 \times 10^5$ N/m	12·95
$k_4 = 1{\cdot}622 \times 10^6$ N/m	$\delta k_4 = -7{\cdot}30 \times 10^4$ N/m	− 4.50
$k_5 = 1{\cdot}622 \times 10^6$ N/m	$\delta k_5 = -9{\cdot}00 \times 10^4$ N/m	− 5.55

TABLE 2

The exact and the perturbation eigenvalues of the system of Figure 1, whose system parameters are given in Table 1. GEVP and SEVP denote the generalized and standard eigenvalue perturbations, respectively

Eigenvalue (1/s^2)	Exact	GEVP, SEVP (% error)
λ_1	28·3624	28·6461 (1·00)
λ_2	221·4155	221·5366 (0·05)
λ_3	595·5432	591·2111 (− 0·73)
λ_4	973·1540	961·8444 (− 1·16)
λ_5	1230·9333	1224·1008 (− 0·56)

on the jth perturbed eigenvector. Table 3 shows the exact modal matrix $[X]$, the perturbed modal matrix $[X_g]$, whose columns are given by equation (14), and the perturbed modal matrix $[X_s]$, whose columns are given by equation (37). The modal matrices are normalized such that the magnitude of the largest element

TABLE 3

The exact model matrix $[X]$, the perturbed modal matrix $[X_g]$, obtained by solving a generalized eigenvalue problem, and the perturbed modal matrix $[X_s]$, obtained by solving a standard eigenvalue problem. The system parameters are identical to those given in Table 1

$$[X]=\begin{bmatrix} 0{\cdot}33062E+00 & -0{\cdot}69549E+00 & 0{\cdot}10000E+01 & 0{\cdot}98046E+00 & 0{\cdot}55702E+00 \\ 0{\cdot}58627E+00 & -0{\cdot}87363E+00 & 0{\cdot}25396E+00 & -0{\cdot}74273E+00 & -0{\cdot}80659E+00 \\ 0{\cdot}77434E+00 & -0{\cdot}43958E+00 & -0{\cdot}91565E+00 & -0{\cdot}11259E+00 & 0{\cdot}10000E+01 \\ 0{\cdot}92315E+00 & 0{\cdot}40007E+00 & -0{\cdot}47081E+00 & 0{\cdot}10000E+01 & -0{\cdot}99003E+00 \\ 0{\cdot}10000E+01 & 0{\cdot}10000E+01 & 0{\cdot}76724E+00 & -0{\cdot}61095E+00 & 0{\cdot}42395E+00 \end{bmatrix}$$

$$[X_g]=\begin{bmatrix} 0{\cdot}32979E+00 & -0{\cdot}68338E+00 & 0{\cdot}10000E+01 & 0{\cdot}97079E+00 & 0{\cdot}55800E+00 \\ 0{\cdot}58689E+00 & -0{\cdot}86246E+00 & 0{\cdot}26595E+00 & -0{\cdot}72228E+00 & -0{\cdot}80607E+00 \\ 0{\cdot}77284E+00 & -0{\cdot}43886E+00 & -0{\cdot}89982E+00 & -0{\cdot}13495E+00 & 0{\cdot}10000E+01 \\ 0{\cdot}92300E+00 & 0{\cdot}40045E+00 & -0{\cdot}47587E+00 & 0{\cdot}10000E+01 & -0{\cdot}95185E+00 \\ 0{\cdot}10000E+01 & 0{\cdot}10000E+01 & 0{\cdot}77271E+00 & -0{\cdot}59936E+00 & 0{\cdot}40165E+00 \end{bmatrix}$$

$$[X_s]=\begin{bmatrix} 0{\cdot}29087E+00 & -0{\cdot}68467E+00 & 0{\cdot}84709E+00 & 0{\cdot}95715E+00 & 0{\cdot}46296E+00 \\ 0{\cdot}53215E+00 & -0{\cdot}86469E+00 & 0{\cdot}68650E+00 & -0{\cdot}73812E+00 & -0{\cdot}10000E+01 \\ 0{\cdot}73057E+00 & -0{\cdot}43967E+00 & -0{\cdot}10000E+01 & -0{\cdot}14647E+00 & 0{\cdot}98016E+01 \\ 0{\cdot}90620E+00 & 0{\cdot}40077E+00 & -0{\cdot}38996E+00 & 0{\cdot}10000E+01 & -0{\cdot}93663E+00 \\ 0{\cdot}10000E+01 & 0{\cdot}10000E+01 & 0{\cdot}99029E+00 & -0{\cdot}57796E+00 & 0{\cdot}57927E+00 \end{bmatrix}$$

within each eigenvector is equal to 1. By inspection, note the excellent agreement between $[X_g]$ and $[X]$ in comparison to $[X_s]$ and $[X]$. It should be emphasized that the only difference between the two perturbed modal matrices is that equation (14) accounts for ε_{jj} while equation (37) assumes $\varepsilon_{jj}=0$.

Finally, it is customary to check the self-compatibility of the perturbed eigenvectors by resorting to the orthogonality characteristics of the normal modes. For the generalized eigenvalue problems, the following orthogonality check may be used:

$$[OR_g]=[X_g]^{\mathrm{T}}[M][X_g], \tag{51}$$

where $[OR_g]$ is an orthogonal matrix, $[M]$ is the system mass matrix, and $[X_g]$ is the perturbed modal matrix normalized such that the diagonal elements of $[OR_g]$ are unity. Similarly, for the standard eigenvalue problems, the following orthogonality check may be utilized:

$$[OR_s]=[X_s]^{\mathrm{T}}[Y_s] \tag{52}$$

where $[OR_s]$ is the orthogonal matrix, $[X_s]$ and $[Y_s]$ are the right and left perturbed modal matrices, respectively, whose columns are given by the perturbed eigenvectors of equations (37) and (38). The modal matrices are normalized such that he diagonal elements of $[OR_s]$ are identically 1.

Theoretically, if the perturbed modal matrices are exact, then the orthogonal matrices correspond to the identity matrix. Since the perturbed modal matrices are

TABLE 4

The orthogonal matrix given by equation (51). The system parameters are identical to shose given in Table 1. The average magnitude of the off-diagonal terms is $\delta_g = 0{\cdot}00345$. The largest magnitude of the off-diagonal elements is 0·00850.

$$[OR_g] = \begin{bmatrix} 0{\cdot}10000E+01 & 0{\cdot}49152E-02 & 0{\cdot}85037E-02 & 0{\cdot}44457E-03 & 0{\cdot}35650E-02 \\ 0{\cdot}49152E-02 & 0{\cdot}10000E+01 & -0{\cdot}10643E-02 & 0{\cdot}29686E-02 & -0{\cdot}37423E-02 \\ 0{\cdot}85037E-02 & -0{\cdot}10643E-02 & 0{\cdot}10000E+01 & 0{\cdot}25199E-02 & -0{\cdot}62670E-02 \\ 0{\cdot}44457E-03 & 0{\cdot}29686E-02 & 0{\cdot}25199E-02 & 0{\cdot}10000E+01 & 0{\cdot}55539E-03 \\ 0{\cdot}35650E-02 & -0{\cdot}37423E-02 & -0{\cdot}62670E-02 & 0{\cdot}55539E-03 & 0{\cdot}10000E+01 \end{bmatrix}$$

TABLE 5

The orthogonal matrix given by equation (52). The system parameters are identical to those given in Table 1. The average magnitude of the off-diagonal terms is $\delta_s = 0{\cdot}03549$. The largest magnitude of the off-diagonal elements is 0·21255.

$$[OR_s] = \begin{bmatrix} 0{\cdot}10000E+01 & 0{\cdot}37158E-01 & 0{\cdot}19208E-02 & -0{\cdot}61104E-03 & -0{\cdot}30838E-03 \\ -0{\cdot}83064E-01 & -0{\cdot}10000E+01 & -0{\cdot}10943E+00 & 0{\cdot}30838E-01 & 0{\cdot}16660E-01 \\ 0{\cdot}14163E-01 & 0{\cdot}21255E+00 & 0{\cdot}10000E+01 & -0{\cdot}44290E-02 & -0{\cdot}26225E-02 \\ 0{\cdot}66516E-03 & 0{\cdot}24683E-01 & 0{\cdot}45968E-02 & 0{\cdot}10000E+01 & 0{\cdot}10168E-02 \\ 0{\cdot}60744E-02 & 0{\cdot}14371E+00 & 0{\cdot}11179E-01 & -0{\cdot}41672E-02 & 0{\cdot}10000E+01 \end{bmatrix}$$

approximate, the magnitudes of the non-zero off-diagonal terms of the orthogonal matrices can be used to pass judgement on the accuracy of the perturbed eigenvectors. Tables 4 and 5 show the orthogonal matrices for the system of Figure 1. Since some off-diagonal magnitudes are larger while others are smaller when comparing one orthogonal matrix to the other, it is difficult to ascertain which perturbation approach leads to a more accurate perturbed modal matrix by simple inspection. In order to make such a comparison quantitatively, an average off-diagonal magnitude is defined as

$$\delta = \frac{1}{N(N-1)} \sum_{r=1}^{N} \sum_{j=1, j \neq i}^{N} |OR(i,j)|, \tag{53}$$

where $|OR(i,j)|$ is the absolute value of the (i,j)th element of the orthogonal matrix, either $[OR_g]$ or $[OR_s]$, depending on which eigenvalue problem is under consideration. A smaller value of δ implies a better perturbed modal matrix. For the set of system modifications of Table 1, the average off-diagonal magnitudes for the generalized and the standard orghogonal matrices are $\delta_g = 0{\cdot}00345$ and $\delta_s = 0{\cdot}03549$, respectively. The magnitudes of the largest off-diagonal elements are $|OR_g(i,j)|_{\max} = 0{\cdot}00850$ and $|OR_s(i,j)|_{\max} = 0{\cdot}21255$. Note that while both approaches lead to the same perturbed eigenvalues, the generalized eigenvector perturbation expression of equation (14) yields a much better approximation to the exact eigenvectors than the standard eigenvector perturbation expression of equation (37). Thus, the coefficient ε_{jj} may have significant effect on the quality of the perturbed eigenvectors.

3. CONCLUSIONS

In this technical note, a comparison is made between the perturbed eigenvalues and eigenvectors obtained by solving a generalized eigenvalue problem and its equivalent standard eigenvalue problem. Analytically and numerically, both approaches lead to the same perturbed eigenvalues. For a generalized eigenvalue problem, the jth perturbed eigenvector expression includes a first order perturbation of the jth unperturbed eigenvector, which is commonly assumed to be zero in the standard eigenvalue problem perturbation formulation. While this correction term to the jth unperturbed eigenvector is of first order, it may have a substantial effect on the accuracy of the perturbed eigenvectors and should not be neglected.

REFERENCES

1. J. C. Chen and B. K. Wada 1977 *AIAA Journal* **15**, 1095–1100. Matrix perturbation for structural dynamic analysis.
2. W. B. Bickford 1987 *International Journal for Numerical Methods in Engineering* **24**, 529–541. An improved computational technique for perturbations of the generalized symmetric linear algebraic eigenvalue problem.
3. T. Inamura 1988 *International Journal for Numerical Methods in Engineering* **26**, 167–181. Eigenvalue reanalysis by improved perturbation.
4. M. S. Eldred, P. B. Lerner and W. J. Anderson 1992 *AIAA Journal* **30**, 1870–1876. Higher order eigenpair perturbations.
5. X. Liu 1997 *International Journal for Numerical Methods in Engineering* **40**, 2019–2035. Perturbation method for reduction of eigenvalue analysis of structures with large stiffnesses and small masses.
6. P. B. Nair, A. J. Keane and R. S. Langley 1998 *AIAA Journal* **36**, 1721–1727. Improved first-order approximation of eigenvalues and eigenvectors.
7. R. G. Parker and C. D. Mote Jr. 1996 *Journal of Vibration and Acoustics* **118**, 436–445. Exact perturbation for the vibration of almost annular or circular plates.
8. D. S. Watkins 1991 *Fundamentals of Matrix Computations*. New York: Wiley.
9. A. Jennings and J. J. McKeown 1992 *Matrix Computation* New York: Wiley.
10. G. H. Golub and C. F. Van Loan 1983 *Matrix Computations*. Baltimore, MD: Johns Hopkins University Press.
11. R. A. Horn and C. R. Johnson 1988 *Matrix Analysis*. Cambridge: Cambridge University Press.
12. L. Meirovitch 1980 *Computational Methods in Structural Dynamics*. The Netherlands: Sijthoff & Noordhoff.
13. C. Pierre and E. H. Dowell 1987 *Journal of Sound and Vibration* **114**, 549–564. Localization of vibrations by structural irregularity.